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# Josephson Effect in a Superconducting Ring*

F. Bloch
*Department of Physics, Stanford University, Stanford, California* 94305
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The Josephson effect in a superconducting ring, interrupted by a barrier, is shown to be the direct consequence of fundamental principles and, hence, to exactly obey the voltage-frequency relation used for a precise determination of $e/h$. Supplementary considerations deal with the role of electron pairing and with additional features such as those found in the presence of two parallel junctions. The special case of a thin ring is used to illustrate the treatment of dynamical properties, including the possible occurrence of hysteresis. A more detailed discussion of the effects due to a potential barrier is presented, followed by a rederivation of Josephson's differential equation which is seen to govern the time dependence of the penetrating flux.

## I. INTRODUCTION

It was pointed out in a preceding letter[1] that the Josephson effect[2,3] can be interpreted as the direct consequence of general principles if one considers the geometry of a superconducting ring interrupted by a barrier. In close analogy to the earlier explanation[4] of quantized flux trapping, the periodic dependence of the current upon the flux through the ring was found sufficient to account for the essential features of the effect. The considerations were restricted to the simplest case where the flux through the ring has a single well-defined value. This situation is encountered if the ring is sufficiently thin to neglect the variation of the flux caused by penetration of the magnetic field into the material or, equivalently, if the shielding by the supercurrent is sufficient to prevent any appreciable penetration. It was shown in particular that one obtains in this case the Josephson relation between voltage and frequency of the current.

A rigorous and general derivation of this relation will be presented in Sec. II, including the previously omitted proof of the periodic flux dependence. Instead of demanding a single value of the total flux, it merely demands this property be satisfied for the part which arises from external sources. Irrespective of any specific assumptions concerning the ring, the Josephson relation will be seen to strictly apply to a reversible process and to refer to the voltage which is induced by a time-dependent external flux through the opening surrounded by the material. Section III deals with additional considerations concerning the effects of pairing, the presence of a barrier, branching, and deviations from reversibility.

A more detailed discussion for the case of a thin ring will be presented in Sec. IV. It shows that the presence of a barrier is essential for the possibility of reversible alternations of the current and includes the criteria for the occurrence of irreversible processes, accompanied by hysteresis, as well as the consideration of dynamic effects. The effects of a barrier are further discussed in Sec. V and extended to take field penetration into account.

While the considerations presented here do not lead to results other than those originally obtained by Josephson, they differ in regard to their derivation and interpretation. Instead of using the specific theories of Ginzburg and Landau, or of Bardeen, Cooper, and Schreiffer, the treatment is based upon some fundamental facts of electrodynamics and quantum mechanics which bear upon the characteristic properties of the superconductive state. It is not possible without specific reference to the microscopic explanation of these properties to evaluate the coefficients which determine the magnitude of the supercurrent. While their change upon the introduction of a barrier will be investigated, there remains a factor of proportionality for which no more than qualitative arguments can be offered within the framework of the present treatment. Since the consideration of a closed ring is essential, the method, furthermore, is not directly applicable to the effects of a voltage across the barrier in an open-ended geometry. The inclusion of this case requires as a separate assumption that the conditions at some distance from the barrier are immaterial to the manifestation of the Josephon effect.

On the other hand, the conclusions reached here are not affected by the unavoidable approximations inherent to the phenomenological or microscopic approach in the theory of superconductivity. They are particularly suited, therefore, to clarify the reasons for the exact validity of the Josephson relation.

## II. GENERAL PROOF OF JOSPEHSON RELATION

The application of a dc voltage $V$ results according to Josephson in an alternating current of frequency

$$\nu = 2\ \mathrm{eV}/h \qquad (1)$$

across the partitioning barrier of a superconductor. Whereas an open-ended geometry permits the use of a battery as voltage source, it is necessary in the case of a superconducting ring to deal with the voltage induced by a time-dependent flux of the magnetic field through the ring. In order to investigate the effect of this voltage, one has to consider that the electromagnetic field arises from the superposition of two parts. One of them is the field contributed by the charged particles in the ring. In addition to the particles themselves, this field is to be regarded as a constituent of the system formed by the ring and is to be described by a separate set of dynamical variables. The other part consists of the field due to external sources which can be arbitrarily controlled so that it enters into the description of the system through a set of adjustable parameters rather than of dynamical variables.

Accordingly, the Hamiltonian representing the total energy of the system in a given external field is to be considered as a function of the dynamical variables which pertain to the particles as well as of those which characterize the field contributed by their charges. While a partial elimination of field variables permits one to express electromagnetic interactions in terms of particle variables, it is neither necessary nor convenient to assume that such an elimination has been carried out. Similarly, it is possible to partly eliminate the variables pertaining to the ions with the result of an effective additional interaction between the conduction electrons. This interaction is essential for the pairing process which leads to the superconductivity of electron systems, and will later be taken into account to obtain the Josephson relation in the form of Eq. (1). The deeper roots of this relation are more evident, however, if the ions are treated as constituent particles of the system on the same basis as the electrons and without explicit reference to their role in the pairing process. It is sufficient at this stage to assign to each of the $N$ particles a definite charge according to its indivdual characterization as an ion or an electron.[5]

Retaining complete generality, the Hamiltonian of the system shall be denoted by

$$\mathcal{H} = \mathcal{H}[\vec{p}_j - e_j \vec{A}(\vec{r}_j)/c, \vec{r}_j] \qquad (2)$$

in its dependence on all the momenta $\vec{p}_j$ and coordinate vectors $\vec{r}_j$ of the particles with charges $e_j\,(j = 1, 2 \cdots N)$ in the presence of the vector potential $\vec{A}(\vec{r})$. Although the notation only emphasizes this particular dependence, a further dependence on spin variables of the particles and on field variables shall be understood without being explicitly indicated. The total magnetic field $\vec{H} = \mathrm{curl}\vec{A}$ derived from the vector potential includes the contribution due to external sources. Writing

$$\vec{A} = \vec{A}_0 + \vec{A}_1, \qquad (3)$$

this contribution shall be given by

$$\vec{H}_1 = \mathrm{curl}\vec{A}_1, \qquad (4)$$

and it is essential that it only enters into the Hamiltonian through the combination of $\vec{A}$ and $\vec{p}_j$ which appears in Eq. (2).

It will be assumed for the purpose of this section that $\vec{H}_1$ is a variable field which vanishes in the whole region $R$ occupied by the ring but contributes the amount $\Phi_1$ to the flux through the opening $O$ (Fig. 1). Such a field is obtained, for example, from the variable current of a long solenoid passing through $O$; any other field $\vec{H}_1$ is equivalent, however, provided that its penetration into the region $R$ is of negligible significance. As the line integral of the external electric field, the applied induced voltage

$$V = -\frac{1}{c}\frac{d\Phi_1}{dt} \qquad (5)$$

has under this condition the same value for any closed path in $R$ which surrounds the opening $O$. The same condition will be seen to lead to an otherwise entirely general periodicity of the free energy and of the current circulating through the ring in their dependence upon $\Phi_1$. The following proof is based upon the method used earlier[4] to explain flux quantization with the difference that it refers to the external flux $\Phi_1$ rather than to the total flux $\Phi$ through the opening.

Inserting $\vec{A}$ from Eq. (3) into the Hamiltonian given by Eq. (2), one obtains the energy levels of the system by solving the equation

$$\mathcal{H}\psi = E\psi \qquad (6)$$

with the condition that the eigenfunction $\psi(\vec{r}_j)$ is

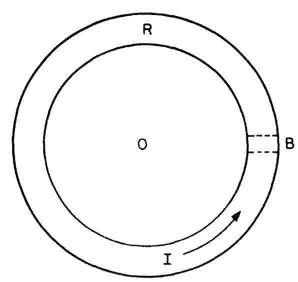


FIG. 1. Superconducting ring $R$ with opening $O$, barrier $B$, and circulating current $I$.

single valued in all particle coordinates $\vec{r}_j$ irrespective of its implicit dependence on other variables. Whereas the magnetic field $\vec{H}_1$ due to external sources can be arbitrary everywhere else, it was assumed that $\vec{H}_1 = 0$ in the region $R$ which contains the particles of the ring, so that in view of Eq. (4) one can write $\vec{A}_1 = \text{grad}\chi_1$ in $R$. Considering that $\psi(\vec{r}_j)$ has to vanish if the location $\vec{r}_j$ of any particle is not inside that region, and noting further that $\vec{p}_j = (h/2\pi i)\,\text{grad}_j$, the gauge transformation

$$\psi(\vec{r}_j) = \psi_0(\vec{r}_j)\exp[2\pi i \sum_j e_j\, \chi_1(\vec{r}_j)/hc] \tag{7}$$

leads from Eqs. (3) and (6) to

$$\mathcal{H}_0\psi_0 = E\psi_0 , \tag{8}$$

where $\mathcal{H}_0$ is the Hamiltonian obtained by letting $\vec{A}_1 = 0$. Since $\psi$ is single valued and since the line integral of $\vec{A}_1$ around $O$ increases $\chi_1$ by the amount

$$\oint \vec{A}_1 \cdot d\vec{s} = \Phi_1 ,$$

$\psi_0$ is multiplied by the factor $\exp(-\,2\pi i\; e_j\,\Phi_1/hc)$ when the particle $j$ is brought around the ring. With all the charges $e_j$ of the particles given as positive or negative integer multiples of the elementary charge $e$, this factor repeats itself whenever $\Phi_1$ changes by the amount

$$\Delta\,\Phi_1 = hc/e\ . \tag{9}$$

The same repetition occurs in the solutions $\psi_0$ and, hence, in the set of energy levels obtained from Eq. (8). Since the latter uniquely determine the partition function $Q$, it follows that the free energy $F = -\,kT\,\ln Q$ is a periodic function of $\Phi_1$ with period $hc/e$. Granting the system under consideration to be invariant against time reversal, it is further seen that $F$ remains unchanged if the sense of rotation around $O$ and, thereby, the sign of $\Phi_1$ is reversed so that $F$ must be an even function of $\Phi_1$. The combination of these two properties permits the free energy of the system to be written as a Fourier series of the general form

$$F = \sum_{n=0}^{\infty} F_n \cos 2\pi n\alpha_1 , \tag{10}$$

where $\quad \alpha_1 = \Phi_1/(hc/e)\,.$ (11)

One should observe that the preceding proof of this important conclusion is entirely based upon some of the most fundamental principles. In fact, its validity requires no more than to accept invariance under a gauge transformation and under time reversal, together with the requirement of single-valued wave functions and the elementary nature of the charge $e$. In particular, the effect of electromagnetic interactions is fully taken into account so that renormalization cannot alter the result. Furthermore, the conclusion holds irrespective of any specific properties of the ring and thus remains valid in the presence of a barrier as a special feature concerning the potential energy of the electrons. Such properties will later have to be taken into account, however, to discuss the hitherto arbitrary magnitude of the coefficients $F_n$.

The free energy refers to the thermal equilibrium of the system at a fixed value of the flux $\Phi_1$, but retains its significance for variable values provided that the variation is sufficiently slow to permit at any instant the establishment of equilibrium. One deals in this case with a reversible process and can use the thermodynamic relation for constant temperature that the rate of change of the free energy represents the work per unit time delivered to the system. Under application of the external voltage $V$ and with a total current $I$ circulating around the ring, one has $dF/dt = I\,V$. With $F$ given as a function of $\Phi_1$ and in view of Eq. (5), it follows that the current is obtained from the free energy by means of the relation

$$I = -\,c\,\frac{dF}{d\Phi_1}\,, \tag{12}$$

which leads through Eqs. (10) and (11) to the result

$$I = \sum_{n=1}^{\infty} I_n \sin 2\pi n\,\alpha_1\ , \tag{13}$$

with $\quad I_n = 2\pi n e\, F_n/h\ .$ (14)

This result has the same general validity as that derived for the free energy and likewise refers not only to constant equilibrium of the system but also to reversible changes under the influence of a time-dependent flux $\Phi_1$.

In particular, the application of a dc voltage corresponds to the linear time dependence

$$\Phi_1 = -\,c\,V t \tag{15}$$

as the result of Eq. (5) for constant $V$ with $\Phi_1 = 0$ chosen at the time $t = 0$. By inserting the corresponding value of $\alpha_1$ from Eq. (11) into Eq. (13), the current

$$I = -\sum_{n=1}^{\infty} I_n \sin(2\pi n e\, V/h)\,t \tag{16}$$

is seen to exhibit a periodic variation with a spectrum of frequencies

$$\nu = neV/h\ , \tag{17}$$

which confirms Josephson's result in a generalized form. Indeed, the preceding proof did not specify the coefficients $I_n$ in Eq. (16) or the integer $n$ in Eq. (17). It requires further considerations, presented in Sec. III, to arrive at the particular choice $n = 2$ for the Josephson relation in the form of Eq. (1). Only the general form of Eq. (17) is needed, on the other hand, to allow the highly precise determination of $e/h$ from a measurement of frequency and voltage.[6] In fact, the fundamental character of this relation demands $n$ to be an exact integer and the knowledge of $e$ and $h$ is sufficiently accurate beforehand to recognize the one and only integer which is compatible with the measurement.

The dc current caused by the simultaneous application of a dc voltage $V$ and an ac voltage $[V'\cos(2\pi\nu' t+\varphi)]$ represents another manifestation of the Josephson effect which can be confirmed with equal generality. In analogy to Eq. (16), one obtains from Eq. (13)

$$I=-\sum_{n=1}^{\infty} I_n \sin\left[\frac{ne}{h}\left(2\pi Vt+\frac{V'}{\nu'}\sin(2\pi\nu' t+\varphi)\right)\right]. \quad (18)$$

A finite time average, and hence, a dc component of the current demands that the relation

$$n'\nu' = ne\,V/h, \quad (19)$$

with exact integers $n$ and $n'$, is satisfied[7] so that this effect provides an equally fundamental method for the precise determination of $e/h$.

## III. SUPPLEMENTARY CONSIDERATIONS

### A. Off-Diagonal Long-Range Order and Pairing

In order to emphasize their fundamental character, the results of Sec. II were derived in such generality as to require no assumption whatever about the nature of conduction so that it seems irrelevant whether or not one deals with a superconductive ring. The basic equations (10) and (13), however, already anticipate the characteristic property of a superconductor to maintain a finite current in the absence of an applied voltage. Indeed, according to Eq. (13), any set of finite coefficients $I_n$ allows the existence of a circulating dc current for constant $\alpha_1$ or $\Phi_1$ and hence, in view of Eq. (5), for a vanishing voltage $V$.

The fact that the coefficients $F_n$ for $n\neq 0$ in Eq. (10), and therefore the coefficients $I_n$ in Eq. (13), can have finite values has been recognized earlier to rest upon a special condition. The resulting flux dependence of free energy and current in the thermal equilibrium of a macroscopic ring requires, in fact, that a superconductor exhibits off-diagonal long-range order (ODLRO)[8] or, equivalently, a singular velocity distribution[9] of the particles responsible for conductivity. Applied to the conduction electrons it was recognized, furthermore, that the exclusion principle prevents this requirement from being met if each electron is considered to move as an individual unit and that the supercurrent has to be attributed to the common motion of electron pairs. This circumstance is taken into account through the replacement of the charge $e$ by $2e$ or, in view of Eq. (11) for $\alpha_1$, through the specification that only coefficients $F_n$ and $I_n$ with even index $n$ can be different from zero since they always appear in combination with $n\alpha_1$.

The preceding conclusion was reached without particular reference to the microscopic origin of the pairing process. Bardeen, Cooper, and Schrieffer[10] have shown that it can be explained by an effective attraction between the electrons which arise from their interaction with the ions. Their theory can, in fact, be used to obtain definite values for the coefficients $F_n$ and $I_n$. While it would be prohibitively difficult to calculate these values from a rigorous treatment of microscopic processes, it must be remembered that any effect of the ions was included in the developments of Sec. II. The restriction to even indices is thus fully consistent with these developments and imposes no further specifications upon the coefficients as long as the ring is considered under quite general conditions.

### B. Current Reduced by Interruption

Among such further specifications, those arising from the interruption by a barrier $B$ (Fig. 1) are of particular importance. An uninterrupted ring has been understood in connection with flux quantization[4] to exhibit a pronounced dependence of the free energy on the flux or, correspondingly, to permit a sizable circulating supercurrent. Any such current is prohibited, on the other hand, in the limit in which the barrier acts as a complete interruption. A continuous variation of the coefficients $I_n$ from relatively large to vanishing values must be expected in the gradual transition from the first of these two extreme cases to the second. In particular, the case of complete interruption can be asymptotically approached by a progressive widening of the barrier, since the transmission coefficient $\theta$ of an electron decreases exponentially with increasing width. It will be shown in Sec. V that $I_n$ is proportional to $\theta^n$ if $\theta\ll 1$ so that for a wide barrier the dominant contribution to the current arises from the term with the smallest even index $n=2$. Neglecting higher terms, one thus obtains from Eq. (13)

$$I=I_2\sin 4\pi\,\alpha_1, \quad (20)$$

and from Eqs. (16) and (17), the Josephson relation in the form of Eq. (1). It is of interest to note that the corresponding expression for the current density, derived by Josephson [J. Eq. (3.11)] from the theory of Ginzburg and Landau, is obtained by replacing $4\pi\alpha_1$ in Eq. (20) by the phase difference of the order parameter on both sides of the barrier.[11]

It is further to be remarked that an increasingly effective potential barrier does not offer the only possibility to cause a gradual reduction of the current. Such a reduction can also be achieved by a progressively narrowing constriction at some location of an otherwise uniform superconducting ring since such a weak link likewise leads to the ultimate prevention of a circulating current. Another practical method consists of the replacement of the barrier by an interrupting layer of normally conducting material. The transition from vanishing to large thickness of this layer similarly results in a gradual reduction of the supercurrent so that in the end merely an Ohmic current is permitted to pass.

### C. Branching

The results of Sec. II can be extended to include the novel features which appear if the ring divides into several branches. For simplicity, it will be assumed that there are only two branches $s$ and $t$, joined by the portion $r$ of the ring (Fig. 2). The external magnetic field will again be considered to vanish in the whole region $R$, composed of the sections $r$, $s$, and $t$, with the difference that besides the flux $\Phi_1$ through the opening $O_1$ it may have a finite flux $\Phi_2$ through the opening $O_2$ surrounded by the two branches.

In order to be single valued, the wave function $\psi$ must remain unchanged whether a particle is brought around the ring passing through the branch $s$ or the branch $t$. Upon elimination of the external

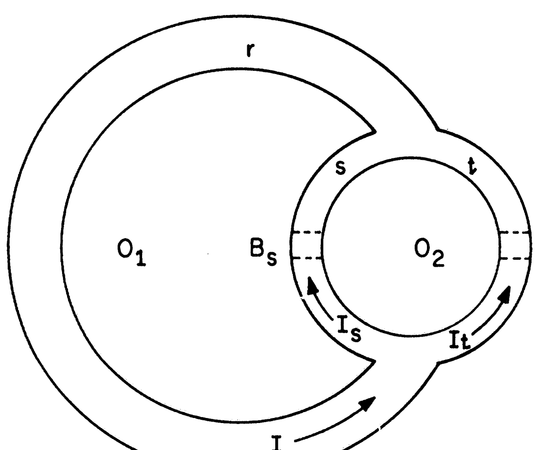


FIG. 2. Superconducting ring with opening $O_1$ which contains two branches $s$ and $t$ with barriers $B_s$ and $B_t$, joined by the portion $r$. The total circulating current $I$ divides into the currents $I_s$ and $I_t$ in the branches, surrounding the opening $O_2$.

vector potential by a gauge transformation and considering that its line integral is equal to $\Phi_1$ or $\Phi_1 + \Phi_2$, depending upon whether one chooses a path of integration through $r$ and $s$ or $r$ and $t$, respectively, the further arguments of Sec. II remain unchanged. It follows that the free energy is a periodic function of $\Phi_1$ as well as of $\Phi_1+\Phi_2$ (or $\Phi_2$) with period $hc/e$. Taking time invariance into account, one thus obtains in analogy to Eq. (10)

$$F=\sum_{m,n} F_{mn}\cos 2\pi[m\alpha_1+n(\alpha_1+\alpha_2)], \tag{21}$$

where $\alpha_2=\Phi_2/(hc/e)$. (22)

The current $I_s$ and $I_t$ through the two branches are obtained from Eq. (12) for the total current

$$I=I_s+I_t \tag{23}$$

and from the analogous relation

$$I_t=-c\frac{dF}{d\Phi_2}, \tag{24}$$

which results from considering the work per unit time in a reversible change of $\Phi_2$. With $F$ given by Eq. (21) one finds that

$$I_s=(2\pi e/h)\sum_{m,n} m F_{mn}\sin 2\pi[m\alpha_1+n(\alpha_1+\alpha_2)], \tag{25}$$

$$I_t=(2\pi e/h)\sum_{m,n} n F_{mn}\sin 2\pi[m\alpha_1+n(\alpha_1+\alpha_2)]. \tag{26}$$

As discussed before, the pairing of conduction electrons demands the summations to be extended only over even indices $m$ and $n$. In the presence of wide barriers $B_s$, $B_t$ with small transmission coefficients $\theta_s$, $\theta_t$ in the branches $s$ and $t$, respectively, $F_{mn}$ is proportional to $\theta_s^m\theta_t^n$. Retaining only the lowest powers with $m=2$, $n=0$ and $m=0$, $n=2$, Eqs. (25) and (26) reduce to

$$I_s=I_{s2}\sin 4\pi\alpha_1, \tag{27}$$

$$I_t=I_{t2}\sin 4\pi(\alpha_1+\alpha_2) \tag{28}$$

as an extension of Eq. (20), obtained for $\alpha_2=0$ with $I_2=I_{s2}+I_{t2}$. For a given time dependence of $\alpha_1$ and, hence, for a given applied voltage according to Eqs. (5) and (11), the maximum of the total current $I=I_s+I_t$ through the circuit is given by

$$|I_2|=(I_{s2}^2+I_{t2}^2+2I_{s2}I_{t2}\cos 4\pi\alpha_2)^{1/2} \tag{29}$$

as a special case of the interference phenomena discussed by Josephson (J. Sec. 3.2.2). The periodic dependence of the maximum current on a static flux $\Phi_2$, provided by a solenoid through the opening $O_2$, has been experimentally established.[12]

### D. Deviations from Reversibility

One is led to an additional consideration by recalling that Eq. (20), as well as the more general equation (13), refers to the ideal limit in which any change within the system is completely reversible. For a given rate of variation of the external parameters, this limit represents the better approximation the more rapidly the system is able to adjust itself to the instantaneous equilibrium. In order to estimate the effect of small deviations, it will be assumed that the adjustment takes a certain relaxation time $\tau$ such that the state of the system at the time $t$ is that of the equilibrium at the slightly earlier time $t-\tau$. Accordingly, Eq. (20) will be modified in the sense that $\alpha_1 = e\Phi_1(t)/hc$ is replaced by $\alpha_1 + \Delta\alpha_1 = e\Phi_1(t-\tau)/hc$. To first order in $\tau$, one thus obtains from Eq. (5) $\Delta\alpha_1 = eV\tau/h$ and to the same order from Eq. (20)

$$I = I_2[\sin 4\pi\alpha_1 + (4\pi eV\tau/h)\cos 4\pi\alpha_1]. \qquad (30)$$

Application of a dc voltage $V$ still leads to an alternating current with the frequency $\nu$ given by Eq. (1) but with a small phase shift for $\nu\tau \ll 1$. The presence of an Ohmic resistance $R_0$ causes another deviation from reversibility and calls for an additional correction proportional to $V$. Combined with Eq. (30) and using the notation

$$(4\pi e\tau I_2)/h = 1/R_2 \ , \qquad (31)$$

one has then

$$I = I_2 \sin 4\pi\alpha_1 + (V\cos 4\pi\alpha_1)/R_2 + V/R_0, \qquad (32)$$

in agreement with the corresponding expression for the current density given by Josephson [J. Eq. (3.10)]. A more pronounced manifestation of irreversibility is associated with the appearance of hysteresis and will be discussed in Sec. IV.

## IV. THIN RING

The rigorous conclusions of Sec. II were reached under the condition that the external magnetic field $\vec{H}_1$ vanishes in the region $R$ occupied by the ring so that its flux through any closed curve around the ring has the same value $\Phi_1$. Irrespective of the magnetic field, this greatly simplifying property of the flux can be used for a sufficiently thin ring and applies in this case not only to the part $\Phi_1$ contributed by external sources but also to the total flux $\Phi$. It can likewise be used for a ring of sizable thickness provided that the penetration of the magnetic field into the region $R$ is of negligible significance. A major exception, to be considered in Sec. V, can arise through the presence of a barrier in such a ring since the magnetic field in the barrier may significantly contribute to the flux through a closed curve and lead to differences of $\Phi$ depending upon where the curve traverses the region of the field. For the purpose of this section, it will be assumed that no such differences are encountered and the abbreviating nomenclature of a "thin" ring is meant to characterize this assumption.

Since the field contributed by the particles has to be taken into account in order to obtain the total flux, it is indicated to separate the energy stored in this field from the total energy of the system. Denoting the corresponding term by $\mathcal{H}''$, one has for the total Hamiltonian $\mathcal{H}$ of Eq. (2)

$$\mathcal{H} = \mathcal{H}' + \mathcal{H}'' \ . \qquad (33)$$

$\mathcal{H}''$ depends only upon the field variables and the dependence of the term $\mathcal{H}'$ upon the particle variables can again be expressed in the form

$$\mathcal{H}' = \mathcal{H}'[\vec{p}_j - e_j\vec{A}(\vec{r}_j)/c, \ \vec{r}_j], \qquad (34)$$

used to indicate this dependence in Eq. (2). Whereas the consideration of quantum effects and a statistical treatment are essential in dealing with the particles, it is permissible with entirely negligible errors to describe the field in classical terms and to ignore its statistical fluctuations. In particular, this allows to regard the total vector potential $\vec{A}(\vec{r})$ as being uniquely determined and to obtain from the eigenvalues of $\mathcal{H}'$ the free energy $F'$ of the particles under the influence of the total magnetic field $\vec{H} = \text{curl}\vec{A}$. Under the conditions of a thin ring, the arguments, used in Sec. II to derive Eq. (10), again apply if one replaces $A_1$ by $\vec{A}$, or $\Phi_1$ by $\Phi = \oint \vec{A}\cdot d\vec{s}$ so that

$$F' = \sum_{n=0}^{\infty} F_n' \cos 2\pi n\alpha \qquad (35)$$

$$\text{with} \quad \alpha = \Phi/(hc/e). \qquad (36)$$

Upon the further replacement of $F$ by $F'$, the derivation of Eq. (12) for the total circulating current $I$ likewise remains valid, thus leading to

$$I = -c\frac{dF'}{d\Phi} \ , \qquad (37)$$

or from Eq. (35) to

$$I(\alpha) = \sum_{n=1}^{\infty} I_n' \sin 2\pi n\alpha \qquad (38)$$

$$\text{with} \quad I_n' = 2\pi ne F_n'/h \qquad (39)$$

in analogy to Eqs. (13) and (14).

While $\mathcal{H}'$ leads to the free energy $F'$ of the particles, the other term $\mathcal{H}''$ in Eq. (33) is responsible for the free energy $F''$ which is stored in their accompanying field. Since thermal properties of the field can be ignored, no distinction between free energy and energy is here required so that the latter can likewise be denoted by $F''$. It is further sufficient for the present purpose to consider only the magnetic field $\vec{H}_0$ caused by the circulating current $I$ through the ring. The energy stored in this field is given by $LI^2/2$ where $L$ is the self-inductance of the ring. Using the relation $\Phi_0 = LcI$ between the current and the flux $\Phi_0$ of the field $\vec{H}_0$, one thus obtains

$$F'' = \Phi_0^2/2Lc^2 \ . \tag{40}$$

Considering that $F'$ appears through Eqs. (35) and (36) as a function of the total flux

$$\Phi = \Phi_0 + \Phi_1 , \tag{41}$$

it is more convient by means of this relation to express $F''$ likewise in terms of $\Phi$ and to write in analogy to Eq. (33)

$$F(\Phi) = F'(\Phi) + F''(\Phi) \ , \tag{42}$$

where $$F''(\Phi) = (\Phi - \Phi_1)^2/2Lc^2 \tag{43}$$

in view of Eqs. (40) and (41).

Although $F(\Phi)$ can in some sense be interpreted as the total free energy of the system, it is important to distinguish this quantity from the actual free energy $F$, given by Eqs. (10) and (11). Whereas $F$ is uniquely determined by the external flux $\Phi_1$, it is seen that, given this part, the total flux $\Phi$ and, hence, $F(\Phi)$ depend upon the variable value of the part $\Phi_0$. The distinction arises from the fact that $F$ refers to thermal equilibrium of the system where $\Phi_0$ has the definite value demanded by the equilibrium current. This value has to be inserted into Eq. (41) for $\Phi$ in order to obtain the argument of the function $F(\Phi)$ at which it is equal to $F$. Equivalently, $F$ is to be characterized as the absolute minimum of this function. In fact, $\Phi$ plays the role of a coordinate and $F(\Phi)$ that of a potential energy for the dynamics of the system so that it will be in equilibrium at the absolute minimum of $F(\Phi)$.[13] It will be necessary, however, to also consider the conditions of stable equilibrium which correspond to other minima of $F(\Phi)$.

With the flux measured in units of $hc/e$ by means of the dimensionless quantities of Eqs. (11) and (36), one obtains from Eqs. (35), (42), and (43)

$$F(\alpha) = \sum_{n=0}^{\infty} F_n' \cos 2\pi n\alpha + h^2(\alpha - \alpha_1)^2/2Le^2 , \tag{44}$$

and an extremum of this function requires in view of Eqs. (38) and (39) that

$$I(\alpha) = q(\alpha - \alpha_1), \tag{45}$$

where $$q = h/Le \ . \tag{46}$$

In addition, one must have

$$\frac{dI}{d\alpha} < q \tag{47}$$

in order to deal with a minimum of $F(\alpha)$ and, hence, with a stable equilibrium. The significance of Eq. (45) for the comparison of flux quantization with the Josephson effect was previously discussed by means of a graphic representation.[1] It was shown, in particular, that the Josephson effect is to be understood as a consequence of a sufficiently reduced current $I(\alpha)$. Indeed, Eqs. (45) and (47) permit in this case only a single solution for $\alpha$ in the vicinity of $\alpha_1$, corresponding to the existence of a single minimum of $F(\alpha)$ and, hence, to a definite equilibrium of the system for every value of $\alpha_1$. The maintenance of this equilibrium represents the condition for a reversible change which was seen in Sec. II to lead from Eq. (13) for the current to the Josephson effect as the result of a linear variation of $\alpha_1$.[14]

A different situation arises, however, if $I(\alpha)$ is large enough to allow several solutions of Eqs. (45) and (47), thus indicating besides the absolute minimum the existence of other minima of $F(\Phi)$. In order to study the transition to this case, it will be assumed that the reduction of the current is caused by a sufficiently wide barrier to result in a transmission coefficient $\theta \ll 1$. In analogy to Eq. (20), only the term with $n = 2$ in the sum of Eq. (38) is then required so that

$$I = I_2' \sin 4\pi\alpha , \tag{48}$$

and from Eqs. (45) and (47)

$$I_2' \sin 4\pi\alpha = q(\alpha - \alpha_1), \tag{49}$$

$$4\pi I_2' \cos 4\pi\alpha < q \ . \tag{50}$$

Since the absence of an external flux $\Phi_1$ obviously permits a stable equilibrium with vanishing total flux $\Phi$, the solution $\alpha = 0$, obtained from Eq. (49) for $\alpha_1 = 0$, must satisfy Eq. (50) for any value of $q$. By going to the limit $q \to 0$, it follows therefore that $I_2' < 0$. Given a finite (negative) value of $I_2'$ and a finite (positive) value of $q$, it now depends upon the ratio $I_2'/q$ whether Eqs. (49) and (50) permit one or several solutions. The transition between these two cases can be seen to occur when $|I_2'/q| = 1/4\pi$ so that one obtains the Josephson effect for $|I_2'/q| < 1/4\pi$.

It will now be assumed, instead, that $|I_2'/q| > 1/4\pi$. Starting with the solution $\alpha = 0$ for $\alpha_1 = 0$, Eqs. (49) and (50) permit a continuous reversible increase of $\alpha$ with increasing $\alpha_1$ until $\alpha$ has reached a value $\alpha_a$ such that $4\pi I_2' \cos 4\pi\alpha_a = q$. At that point, the system is in a state of indifferent equilibrium but the same value of $\alpha_1$ permits one or several solutions of stable equilibrium, depending upon the magnitude of $|I_2'/q|$. The transition to such a new equilibrium with $\alpha$ at a new value $\alpha_b$ will take place in an irreversible process whereupon $\alpha$ again increases reversibly with a further increase of $\alpha_1$. This alternation between reversible processes repeats itself periodically upon a

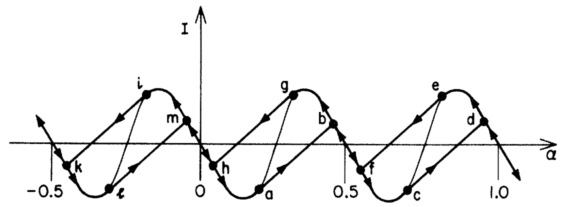


FIG. 3. Circulating current $I$ in a thin ring versus flux $\alpha$, measured in units $hc/e$. The straight sections $(a, b)$, $(c, d)$, $(l, m)$ and $(e, f)$, $(g, h)$, $(i, k)$ with slope $q$ indicate irreversible transitions at the values of the external flux $\alpha_1$ in units of $hc/e$, given by their intercept with the $\alpha$ axis, for slowly increasing and decreasing values of $\alpha_1$, respectively.

monotonic increase of $\alpha_1$ and an analogous alternation with the accompanying hysteresis takes place upon the reversed change of $\alpha_1$. Both are illustrated in Fig. 3 with $|I_2'/q|$ chosen such that for each irreversible transition only a single final equilibrium is available and Fig. 4 represents the corresponding variation of $\alpha$ with $\alpha_1$. It is to be noticed that successive irreversible transitions occur upon an increment or decrement of $\alpha_1$ by the amount $\frac{1}{2}$ or according to Eq. (11) upon a change of the external flux $\Phi_1$ by the amount of the flux quantum

$$\Phi^* = hc/2e \ . \tag{51}$$

The observation of these transitions thus serves their well-known use for the measurement of small changes of a magnetic field.

Considering ever larger values of $|I_2'/q|$, irreversible transitions can end up in an increasing number of stable equilibria and it then depends upon the dynamics of the system which of them will actually be reached. Accordingly, the change of the flux through the ring, undergone in such a transition, can assume an increasing number of values and it can be seen that these values amount the more closely to integer multiples of the flux quantum the larger $|I_2'/q|$. The experiments of Silver and Zimmerman[15] clearly demonstrate such irreversible transitions under different conditions which are essentially equivalent to different magnitudes of $I_2'$ although the reduction of the current is achieved by a variable weak link instead of a barrier.

In order to investigate dynamic effects, it is necessary to add to the magnetic part of the field energy the contribution stored in the electric field. With the voltage due to the flux $\Phi_0$ given by

$$V_0 = -\frac{1}{c}\frac{d\Phi_0}{dt} \ , \tag{52}$$

this contribution can be written in the form $\frac{1}{2}CV_0^2$, where $C$ is the effective capacity of the circuit.[16]

Equation (40) for the energy stored in the field of the ring is thus replaced by

$$F'' = C\left(\frac{d\Phi_0}{dt}\right)^2 \Big/ 2c^2 + \Phi_0^2/2Lc^2 \ . \tag{53}$$

The rate of change of this energy is given by $-V_0 I$, the work per unit time performed by the current against the voltage $V_0$. Therefore,

$$\frac{dF''}{dt} = \frac{1}{c}\frac{d\Phi_0}{dt} I \ ,$$

and with $F''$ from Eq. (53)

$$(1/c)(C\ddot{\Phi}_0 + \Phi_0/L) = I \ . \tag{54}$$

In order to account for damping effects, an Ohmic term will be added to the expression for the current of Eq. (48) so that one has for the right side of Eq. (54)

$$I = I_2' \sin 4\pi\alpha + V/R_0 . \tag{55}$$

With $V = -(1/c)(d\Phi/dt)$ and using Eqs. (11), (36), (41), and (46), one thus obtains from Eqs. (54) and (55)

$$q(LC\ddot{\alpha} + L\dot{\alpha}/R_0 + \alpha - LC\ddot{\alpha}_1 - \alpha_1) = I_2' \sin 4\pi\alpha \tag{56}$$

as a differential equation for the total flux $\alpha(t)$ at arbitrarily varying external flux $\alpha_1(t)$, both measured in units of $hc/e$.

If $\alpha_1$ varies sufficiently slow to neglect its derivatives, Eq. (56) reduces to Eq. (49) provided that the derivative of $\alpha$ can be assumed to be like-

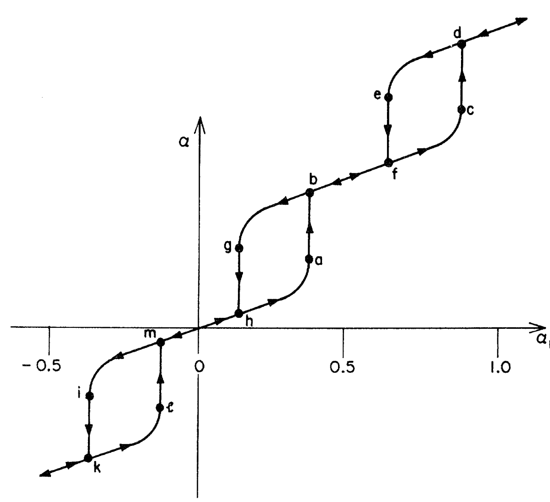


FIG. 4. Total flux $\alpha$ versus external flux $\alpha_1$, both measured in units $hc/e$. The plot is obtained from Fig. 3 by the following construction: A straight line with slope $q$ through a point with the abscissa $\alpha$ on the curve in Fig. 3 intercepts the $\alpha$ axis at the corresponding value of $\alpha_1$.

wise negligible. The single solution obtained for $|I_2'/q| < 1/4\pi$ is compatible with this assumption since it can be shown to be stable against small perturbations. In view of the preceding considerations, Eq. (56) thus correctly describes the Josephson effect under the appropriate conditions. For $|I_2'/q| > 1/4\pi$, the derivates of $\alpha$ are no longer negligible, however, when the solution of Eq. (49) has reached the value, previously discussed, at which equilibrium ceases to exist. Indeed, a small deviation from this value can be seen to first build up exponentially with the subsequent time dependence to be obtained by integration of Eq. (56). Because of the damping term, proportional to $1/R_0$, $\alpha$ will finally reach a new value corresponding to stable equilibrium and the magnitude of the damping coefficient determines which among several such equilibria will actually be established.

The application of an external dc voltage corresponds to a linear variation of $\alpha_1$ and, hence, to the absence of the term with $\ddot{\alpha}_1$ in Eq. (56). The solutions can in this case be demonstrated by the mechanical analog of a pendulum with viscous friction, connected by an elastic spiral spring to a coaxial shaft which rotates with constant angular velocity $\omega$.[17] Denoting the angular deviation of the pendulum from the vertical by $\beta$ and the angle of rotation of the shaft by $\beta_1$ with the spring unloaded for $\beta_1 = \beta$, $4\pi\alpha$ and $4\pi\alpha_1$ are in the analog to be replaced by $\beta$ and $\beta_1$, respectively, so that a full turn of these angles represents in view of Eqs. (11) and (36) an increase of the corresponding flux by the flux quantum $\Phi^*$ of Eq. (51). It can further be seen that the correspondence of the applied dc voltage $V$ to the frequency $\nu = \omega/2\pi$ with which the shaft rotates is that of the Josephson relation given in Eq. (1). The conditions for the Josephson effect and for the irreversible processes, considered above, are reproduced by choosing for the pendulum a relatively small and large mass, respectively, with the damping term adjusted by means of the coefficient of friction.[18]

## V. EFFECTS OF A POTENTIAL BARRIER

It was remarked in Sec. III B that continuity demands a gradual reduction of the circulating current with increasing width of a barrier. In order to investigate this effect in greater detail, it is sufficient to represent the barrier by the constant potential energy $U$ of an electron between two parallel planes, separated by the width $w$. In the vicinity of the barrier, a coordinate system will be used with the $z$ axis perpendicular to the two boundary planes, located at $z = 0$ and $z = w$ (Fig. 5).

Starting with the case of a thin ring, discussed in Sec. IV, the magnetic field in the region $R$, including the barrier, will be assumed to be negligibly small. The total vector potential in this region can thus be written as $A = \text{grad}\chi$ and is eliminated from the particle Hamiltonian $\mathcal{H}'$ of Eq. (34) by means of the gauge transformation

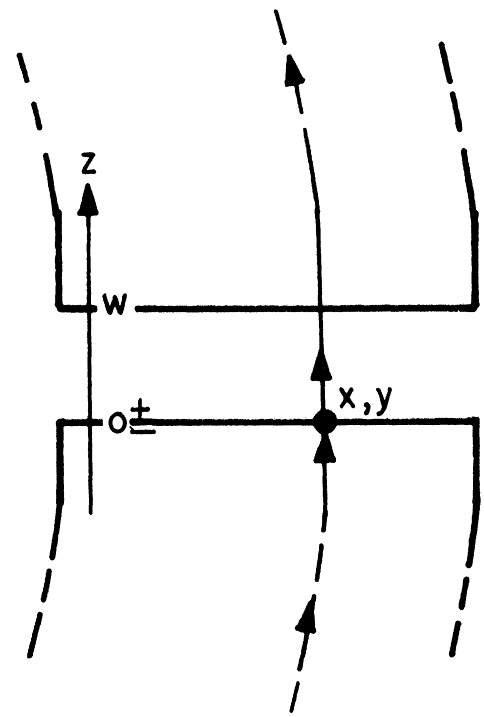


FIG. 5. Vicinity of a barrier with width $w$. In the absence of field penetration, the flux can be obtained from the line integral of the vector potential around the ring starting from a point with coordinates $x$, $y$ on the positive side of the plane $z = 0$ and ending at the same point on the negative side.

$$\psi(\vec{r}_j) = \psi_0(\vec{r}_j) \exp[2\pi i \sum_j e_j \chi(\vec{r}_j)/hc] \quad . \tag{57}$$

Analogous to Eq. (8), one has then

$$\mathcal{H}_0' \psi_0 = E' \psi_0 , \tag{58}$$

where $\mathcal{H}_0'$ is obtained from $\mathcal{H}'$ by letting $\vec{A} = 0$. The notation $E'$ instead of $E$ is used to distinguish the eigenvalues of $\mathcal{H}'$ from those of the total Hamiltonian $\mathcal{H}$. Considering a particular electron and demanding $\psi$ to be single valued, it follows with $\alpha$ given by Eq. (36) that $\psi_0$ is to be multiplied by $e^{-2\pi i\alpha}$ when the electron is brought around the ring. This can be done by starting from a point $x$, $y$ on the plane $z = 0$ in the direction of positive $z$ so that after going around the ring one returns to the same point from the side of negative $z$. The two sides of the plane shall be indicated by $z = 0_+$ and $z = 0_-$. Omitting the dependence of $\psi_0$ on all variables except the $z$ coordinate of the electron in the vicinity of the barrier, one can write $\psi_0 = \psi_0(z)$ so that one has to demand

$$\psi_0(0_-) = e^{-2\pi i\alpha}\,\psi_0(0_+)\,. \tag{59}$$

Since instead of the plane $z=0$, any neighboring plane could equally well have been chosen, the same relation has to hold for the $z$ derivative with the result

$$\left.\frac{\partial\psi_0}{\partial z}\right|_{0-} = e^{-2\pi i\alpha}\left.\frac{\partial\psi_0}{\partial z}\right|_{0+} \tag{60}$$

to be noted for the following purposes.

The potential energy $U$ in the barrier will be considered sufficiently large to cause the dominant $z$ dependence of $\psi_0$ for $0<z<w$ so that in this range

$$\psi_0(z) = ae^{\kappa z} + be^{-\kappa z}$$

with $a$ and $b$ independent of $z$ and with

$$\kappa = (2m\,U/\hbar^2)^{1/2}\,. \tag{61}$$

$\psi_0$ and $\partial\psi_0/\partial z$ at the two boundaries of the barrier are thus connected by the relations

$$\left.\left(\kappa\psi_0 + \frac{\partial\psi_0}{\partial z}\right)\right|_{0+} = \left.\left(\kappa\psi_0 + \frac{\partial\psi_0}{\partial z}\right)\right|_{w} e^{-\kappa w}\,,$$

$$\left.\left(\kappa\psi_0 - \frac{\partial\psi_0}{\partial z}\right)\right|_{w} = \left.\left(\kappa\psi_0 - \frac{\partial\psi_0}{\partial z}\right)\right|_{0+} e^{-\kappa w}\,,$$

or in view of Eqs. (59) and (60) by

$$\left.\left(\kappa\psi_0 + \frac{\partial\psi_0}{\partial z}\right)\right|_{0-} = \left.\left(\kappa\psi_0 + \frac{\partial\psi_0}{\partial z}\right)\right|_{w} e^{-\kappa w - 2\pi i\alpha}\,, \tag{62}$$

$$\left.\left(\kappa\psi_0 - \frac{\partial\psi_0}{\partial z}\right)\right|_{w} = \left.\left(\kappa\psi_0 - \frac{\partial\psi_0}{\partial z}\right)\right|_{0-} e^{-\kappa w + 2\pi i\alpha}\,. \tag{63}$$

Considering that the side $z=0_-$ of the plane $z=0$ is reached from the plane $z=w$ by going around the ring, the relation of Eqs. (62) and (63) represents boundary conditions for the solutions of Eq. (58) in the open-ended part of the region $R$ which remains upon exclusion of the part occupied by the barrier. These conditions must be satisfied for each electron at all points $x, y$ on the two boundary planes and determine the complete set of eigenvalues $E'$ to be admitted in Eq. (58). In the limit $\kappa\to\infty$ of an infinitely high barrier, they reduce to the familiar condition that the wave function has to vanish at the boundaries.

Generally, the barrier width as well as the total flux $\Phi$ through the ring enter only through the exponentials on the right sides of Eqs. (62) and (63). Under otherwise given conditions, the eigenvalues $E'$ therefore only depend upon these expontials and result in the free energy $F'$ of the particles as a function of $\theta e^{2\pi i\alpha}$ and $\theta e^{-2\pi i\alpha}$, where

$$\theta = e^{-\kappa w} \tag{64}$$

is the transmission coefficient. Expanding in powers of $\theta$, one has

$$F'(\theta e^{2\pi i\alpha}, \theta e^{-2\pi i\alpha}) = \sum_{l,m=0}^{\infty} c_{lm}\, e^{2\pi i(l-m)\alpha}\theta^{l+m}$$

or
$$F' = \sum_{n=-\infty}^{+\infty} e^{2\pi i n\alpha} \sum_{\mu=0}^{\infty} d_{n\mu}\,\theta^{|n|+2\mu} \tag{65}$$

with fixed coefficients $c_{lm}$ or $d_{n\mu}$. Considering finally that $F'$ must be a real even function of $\alpha$, one obtains Eq. (35) with

$$F_0' = \sum_{\mu=0}^{\infty} d_{0\mu}\,\theta^{2\mu} \tag{66}$$

and
$$F_n' = 2\theta^n \sum_{\mu=0}^{\infty} d_{n\mu}\,\theta^{2\mu} \tag{67}$$

for $n>0$. For a sufficiently wide barrier with correspondingly small transmission coefficient $\theta$, only the terms with $\mu=0$ need to be retained so that $F_n' \sim \theta^n$ and from Eq. (39) $I_n' \sim \theta^n$. With pairing taken into account, the dominant contribution to the sum in Eq. (38) arises in this case from the term with $n=2$ so that one obtains Eq. (48) for the current with $I_2' \sim \theta^2$. Considering that $\theta$ represents the transmission coefficient for a single electron, it is plausible that the current will be proportional to $\theta^2$ since it is maintained by the simultaneous tunneling of electron pairs through the barrier. Neglecting higher terms in $\theta$, it can further be shown that[19] $I_n = I_n'$ so that the results obtained above lead at the same time to the properties of the coefficients $I_n$ used in Sec. III B.

The preceding discussion was based upon the assumption that the penetration of the magnetic field into the ring is negligible. Because of the Meissner effect, this assumption can be safely made, except in the vicinity of the barrier since the current across the barrier may be too small to provide sufficient shielding. One deals in this case with a magnetic field not only inside the barrier but also in the region extending beyond its boundary planes to a distance comparable to the London penetration depth. It will be assumed that this region lies between the planes $z=-\delta$ and $z=w+\delta$ (Fig. 6), such that the magnetic field at greater distances can be considered to be vanishingly small. In accordance with the corresponding treatment by Josephson (J. Sec. 3.1) it will be further assumed that the sideways dimensions of the barrier are sufficiently large compared to $w$ and $\delta$ to neglect edge effects and that the penetrating field has a vanishing $z$ component.

Denoting the part of the vector potential responsible for the flux through the opening $O$ again by grad$\chi$ and the part due to the penetrating field by $\vec{a}$, one has thus inside the region $R$

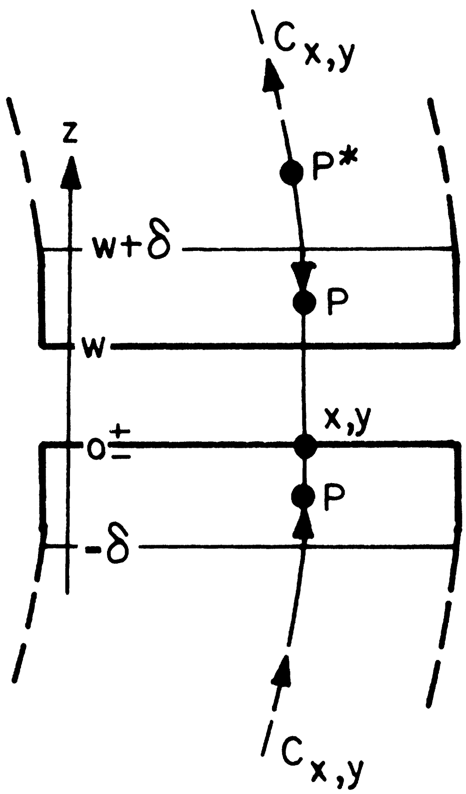


FIG. 6. Vicinity of a barrier. The field is assumed to penetrate into the region between the planes $z=-\delta$ and $z=w+\delta$. In this region, the curve $C_{x,y}$ around the ring is chosen to run parallel to the $z$ axis and to pass the plane $z=0$ at a point with coordinates $x$, $y$. One has to consider line integrals of the vector potential along this curve, starting at a fixed point $P^*$ outside the region of penetration and ending on the variable point $P$. Two alternative positions of $P$ are chosen to indicate how they are to be reached without passing through the plane $z=0$.

$$\vec{A}=\text{grad}\chi+\vec{a}, \tag{68}$$

where $a_x=a_y=0$ and where $a_z\neq 0$ only for $-\delta<z<w+\delta$. In contrast to the case of negligible penetration, it is not possible to rigorously eliminate the vector potential by a gauge transformation. An approximate elimination is achieved, however, by replacing $\chi$ in Eq. (57) by $\chi+\chi_a$, where

$$\chi_a=\int_{P^*}^{P}\vec{a}\cdot d\vec{s} \tag{69}$$

is defined as a suitably chosen line integral within $R$ from a fixed point $P^*$ in the region of vanishing magnetic field to a variable point $P$. The path of integration is chosen to pass through the plane $z=w+\delta$ or $z=-\delta$ if $P$ is a point between these planes with $z>0$ or $z<0$, respectively, and to traverse the intermediate space in both cases parallel to the $z$ axis. $\chi_a$ thus becomes a uniquely defined function of the coordinates of $P$ which satisfies the relation $\partial\chi_a/\partial z=a_z$. Whereas $a_x=a_y=0$, one finds, however, that

$$\frac{\partial\chi_a}{\partial x}=-\int H_y dz \quad\text{and}\quad \frac{\partial\chi_a}{\partial y}=\int H_x dz,$$

where $H_x$ and $H_y$ are the components of the penetrating field $\vec{H}=\text{curl}\vec{a}$. Consequently, the vector condition $\vec{a}=\text{grad}\chi_a$, required for elimination of the vector potential $\vec{A}$, is satisfied only in regard to its $z$ component.

It is permissible, nevertheless, to maintain the validity of Eq. (58), based upon complete elimination of $\vec{A}$, by assuming that the $x$ and $y$ derivatives of $\chi_a$ are sufficiently small to neglect their appearance in the expression for the transformed Hamiltonian. Considering that they are obtained by integrating the components of $\vec{H}$ over intervals of $z$ no larger than $w+\delta$, this assumption imposes no severe limitation upon the magnitude of the penetrating field. It can be shown to merely imply a negligible effect of the Lorentz force caused by field penetration and corresponds to the assumption, made in Josephson's treatment, that the magnetic field does not appreciably affect the magnitude of the order parameter. Although the $x$ and $y$ derivatives of $\chi_a$ are thus assumed to be sufficiently small, it should be noted that this does not exclude an appreciable variation of $\chi_a$ over the relatively large sideways dimensions of the barrier.

By extending the integral in Eq. (69) to the point $P$ with coordinates $x$, $y$, $0_+$ from the side $z>0$ and $x$, $y$, $0_-$ from the side $z<0$, it is seen that the corresponding values of $\chi_a$ differ by the contribution

$$\oint\vec{a}\cdot d\vec{s}$$

of the penetrating field to the flux through a closed curve $C_{x,y}$ around the ring. This contribution arises only from the vicinity of the barrier which is to be traversed on a straight line parallel to the $z$ axis with coordinates $x$ and $y$. In view of Eq. (68), the total flux

$$\oint\vec{A}\cdot d\vec{s}$$

through $C_{x,y}$ is therefore given by

$$\Phi(x,y)=\Delta\chi+\int_{-\infty}^{+\infty}a_z(x,y,z)dz, \tag{70}$$

where $\Delta\chi$ represents the contribution of the field through $O$ and is independent of $x$ and $y$. The fact that the wave function $\psi$ must be single valued, combined with the effect of the barrier, thus leads again to Eqs. (62) and (63) as boundary conditions for $\psi_0$. As an important difference from the preceding case, however, $\alpha$ is no longer a constant but a function of $x$ and $y$, given by

$$\alpha(x,y)=\frac{\Phi(x,y)}{hc/e}\ . \tag{71}$$

Instead of a function of the variables $\theta e^{\pm 2\pi i\alpha}$, $F'$ becomes a functional of $\theta e^{\pm 2\pi i\alpha(x,y)}$ or, considering pairing, of $\theta^2 e^{\pm 4\pi i\alpha(x,y)}$. The coefficients of an expansion in powers of $\theta^2$ are here multiple integrals of the form

$$\int K(x_1y_1, x_2y_2\cdots)\exp(4\pi i[\pm\alpha(x_1y_1)\pm\alpha(x_2y_2)\pm\cdots])$$
$$\times\, dx_1dy_1dx_2dy_2\cdots$$

so that it is not possible, in general, to express $F'$ as the surface integral of a definite free energy per unit area of the barrier. This is possible, however, if the transmission coefficient is sufficiently small so that all higher terms in $\theta^2$ can be neglected. As a real quantity which must be even in $\alpha$, one has in this case

$$F' = F_0' + f_2' \int \cos 4\pi\alpha(x,y)\, dx\, dy\,, \tag{72}$$

where $f_2' = K\theta^2$

and where, because of the omission of edge effects, the more general kernel $K(x,y)$ is replaced by the constant $K$.

The fact that $a_z$ differs from zero only in the vicinity of the barrier shall be formulated by writing

$$a_z = g(x,y)h(z)\,, \tag{73}$$

where $h(z)$ is essentially constant for $0<z<w$ and proportional to $e^{-|z|/\lambda_0}$ for $z>w$ and $z<0$. $\lambda_0$ stands here for the London penetration depth so that $\delta$, while small compared to the sideways dimensions of the barrier, has to be chosen several times larger than $\lambda_0$. It follows then from Eq. (70) that

$$\Phi(x,y) = \Delta\chi + g(x,y)\int_{-\infty}^{+\infty} h(z)dz\,. \tag{74}$$

In analogy to the derivation of Eq. (13), one obtains relations for the current density $\vec{j}$ in the vicinity of the barrier by letting $\vec{A}$ and thereby $\Delta\chi$ and $g(x,y)$ depend upon the time. The isothermal work per unit time done upon the system is given by

$$\frac{dF'}{dt} = \int (\vec{j}\cdot\vec{E})\, d\tau\,, \tag{75}$$

where $\vec{E} = -[\mathrm{grad}\varphi + (1/c)(\partial\vec{A}/\partial t)]$ represents the electric field and $\varphi$ the scalar potential. With $\mathrm{div}\vec{j}=0$ and using Eqs. (68), (71)–(74), comparison of the terms with $\partial\Delta\chi/\partial t$ and $[\partial g(x,y)]/dt$ on both sides of Eq. (75) can be shown to yield the relations

$$\int j_z\, dx\, dy = j_2' \int \sin 4\pi\alpha\, dx\, dy \tag{76}$$

and

$$[\int h(z) j_z\, dz]/[\int h(z)\, dz] = j_2' \sin 4\pi\alpha\,, \tag{77}$$

respectively,[20] where

$$j_2' = 4\pi f_2' e/h\,.$$

These relations are not independent since multiplication with $h(z)$ and integration over $z$ on both sides of Eq. (76) gives the same result as integration over $x$ and $y$ on both sides of Eq. (77). The left side of Eq. (76) represents the total circulating current $I$ and includes Eq. (48) as the special case of constant $\alpha$ with $I_2'$ obtained by multiplying $j_2'$ with the area of the barrier. Since $I_2'$ was found to be negative it follows that $j_2' < 0$.

In order to arrive at a differential equation for $\alpha$, one has to take the line integral over the closed curve $C_{x,y}$ on both sides of the Maxwell equation $\mathrm{curl}\vec{H} - (1/c)(\partial\vec{E}/\partial t) = 4\pi\vec{j}/c$. Noting that

$$\oint (\vec{E}\cdot d\vec{s}) = -\frac{1}{c}\frac{\partial\Phi}{\partial t}\,,$$

$\vec{H} = \mathrm{curl}\vec{a}$, and that $\vec{j}$ as well as $\vec{a}$ differ from zero only in the vicinity of the barrier where $C_{x,y}$ is parallel to the $z$ axis, one finds that

$$-\nabla^2\int a_z\, dz + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{4\pi}{c}\int j_z\, dz\,, \tag{78}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

In view of Eq. (77), the right side of Eq. (78) can be expressed in terms of $\alpha$ if one defines an effective width $d$ of the barrier by

$$d = [\int h(z)dz][\int j_z\, dz]/[\int h(z) j_z\, dz]\,. \tag{79}$$

It reduces to $d=w$ if $\lambda_0 \ll w$ and if $h(z)$ is constant for $0<z<w$, but otherwise depends upon how far $a_z$ and $j_z$ extend beyond the barrier. Since $\Delta\chi$ is independent of $x, y$, one obtains from Eqs. (70), (71), (77), (78), and (79)

$$\nabla^2\alpha - \frac{1}{c^2}\frac{\partial^2\alpha}{\partial t^2} = \frac{1}{4\pi\lambda^2}\sin 4\pi\alpha\,, \tag{80}$$

where $\lambda = (-hc^2/16\pi^2 j_2' ed)^{1/2}$

in agreement with the corresponding result derived by Josephson[21] [J. Eqs. (3.12), (3.14)].

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[5]One may go further and include in the particles the

neutrons and protons constituting the nucleus of the ions or even any virtually present mesons. The following proof merely requires that all charges are positive or negative integer multiples of the elementary charge $e$.

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[7]The expansion of the right side of Eq. (18) in powers of $V'$ shows that the current consists of a sum of terms proportional to $\exp\{2\pi i t[\pm(neV/h)\pm n'(\nu')]\}$ with integer values $n'$. Equation (19) expresses the condition that a conjugate complex pair of these terms has a finite time average.

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[9]F. Bloch, Phys. Rev. 137, 787 (1965).

[10]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[11]Applied to a closed ring with single-valued order parameter, the equivalent relation between phase difference and enclosed flux can be shown, conversely, to be a consequence of the Ginzburg-Landau theory. It should be observed, however, that the derivation of Eq. (20), presented here, does not hinge upon this theory.

[12]R. C. Jaklevic, J. J. Lambe, A. H. Silver, and J. E. Mercereau, Phys. Rev. Letters 12, 274 (1964).

[13]The exact definition of $F$ by means of the partition function $Q$ requires the consideration of the integral $\int\exp[-F(\Phi)/kT]\,d\Phi$. As a macroscopic quantity, $F(\Phi)$ causes a sharp decrease of the integrand as soon as $\Phi$ deviates even slightly from the value at which $F(\Phi)$ has its absolute minimum $F_{\min}$ so that $Q\sim\exp[-F_{\min}/kT]$.

[14]The dependence of the equilibrium current on $\alpha_1$, given by Eq. (13), is obtained by inserting the solution of Eq. (45) for $\alpha$ into Eq. (38). This allows one to express the coefficients $I_n$ in terms of the coefficients $I_n'$; in particular, one finds for a sufficiently small current that $I_n \cong I_n'$ since it follows in this case from Eq. (45) that $\alpha\cong\alpha_1$.

[15]A. H. Silver and J. E. Zimmerman, Phys. Rev. 157, 317 (1967).

[16]In the presence of a barrier, the electric field may be assumed to be negligible everywhere else so that $C$ represents in that case the capacity of the barrier acting as a condensor.

[17]H. E. Rorschach and J. T. Carter (unpublished). The analogy of a pendulum has been previously noted by P. W. Anderson, in *Lectures on the Many-Body Problem, Ravello*, 1963, edited by E. R. Caianello (Academic, New York, 1964), Vol 2, p. 126.

[18]With the mass $M$ of the pendulum concentrated at a distance $D$ from the axis, one finds the equation of motion $MD\ddot{\beta}+K\dot{\beta}+f(\beta-\beta_1)=-Mg\sin\beta$, where $f$ and $K$ are proportional to the spring constant and the coefficient of friction, respectively. The analog is verified by comparison with Eq. (56) for $\ddot{\alpha}_1=0$ whereby the constancy of $\dot{\alpha}_1$ corresponds to letting $\beta_1=-\omega t$. In particular, the ratio $I_2'/q$ is seen to be replaced by $-Mg/4\pi f$ so that increasing values of $L$ (with $R_0$ proportional to $L$) and, hence, of $|I_2'/q|$ are indeed reproduced by choosing an increasingly larger mass $M$ of the pendulum.

[19]See Ref. 14.

[20]The proof of Eq. (76) requires consideration of the integral $\int\mathrm{div}[\vec{j}\,(\partial\chi/\partial t)]\,d\tau$. Dividing the connected region $R$ by a plane $z=\text{const}$ and applying the Gauss theorem, one obtains $(\partial\Delta\chi/\partial t)\int j_z\,dx\,dy$, since the values of $\chi$ on the two sides of the plane (e.g. the sides $0_+$ and $0_-$ of the plane $z=0$) differ by the amount $\Delta\chi$.

[21]Analogous to the remark in Sec. III B, the correspondence requires the replacement of $4\pi\alpha$ by the difference of the order parameter on both sides of the barrier. In Josephson's notation, the positive constant $-j_2'$ is further replaced by $j_1$ and the identification of $v$ with $c$ follows from his Eq. (3.13) if one inserts for the capacity $C$ per unit area of the barrier that of a plane condensor with the two plates separated by the effective distance $d$. The introduction of $d$ by Josephson in Sec. (3.1) is essentially equivalent to the definition given here through Eq. (79).